"equilibrium" phase velocity of the longitudinal sound waves.

When the internal mass moment is neglected (in the limit as $\gamma \to \infty$) from (4.3) it follows that in the transverse waves the electromagnetic quantities $(\pi, \mathbf{H}, \mathbf{E})$, and in the longitudinal waves the mechanical quantities $(\mathbf{v}, \rho)$ are the only ones perturbed.

In addition to the two types of weak perturbations discussed here, we also have a solution of (4.5) corresponding to the case when $\omega = 0$. This wave does not propagate through space and represents an arbitrarily small deviation in the entropy distribution from its value in the equilibrium state. From (4.5) it follows that in the entropic wave $\pi = 0$, $\mathbf{H} = \mathbf{E} = 0$, $\mathbf{v} = 0$ and the only non-zero perturbations are those of density and entropy.

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NON-LINEAR EQUATIONS OF THE DYNAMICS OF AN ELASTIC MICROPOLAR MEDIUM*

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The non-linear equations for a continuous elastic medium with three additional degrees of freedom associated with local rotation, are considered. Such an elastic medium is called micropolar /1/. The existence of an elastic potential is proposed for it; thermal effects are neglected.

The purpose of this paper is to study certain qualtiative properties of the equations that are closely associated with the concept of hyperbolicity. The complete set of equations is represented as a system of local conservation laws, closed by finite relationships yielding the rheology of the material. The possibility of such a representation is based /2, 3/ on the fact that the gradients of the particle displacment and angle of rotation are used as a measure of the deformation. Local conservation laws for the compatibility of the strain and velocity fields of fairly simple structure are formulated.

The velocities of propgation of characteristic surfaces are studied for the dynamic equations for the general case of the material under consideration. The existence of real velocities, the necessary condition for hyperbolicity, results in a constraint on the form of the elastic potential function, which is an analog of the *SE*-inequality /4/ in the classical theory of non-linearly elastic media.

The system of non-linear equations being studied is reduced to symmetric form by replacing the vector of the solution. The necessary condition for such a transformation /5/ is the existence of an additional energy conservation law that follows from the system under consideration. The symmetric form of the equations enables us to formulate the sufficient condition for hyperbolicity - the condition of convexity of the elastic potential in its arguments. An estimate is obtained for the growth of the solutions of the Cauchy problem and the ensuing uniqueness theorem. The presence of the symmetric form of the system enables a general form to be obtained for the transport equation that governs the rate of change of a weak discontinuity along a bicharacteristic.

1. Fundamental equations.

Let $\mathbf{X}$ be the radius-vector of a material particle of a body in the reference configuration $\varkappa$. We assume that the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ and

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and the rotation vector $\varphi = \varphi(X, t)$ are defined in $\varkappa$. This assumption corresponds to a micropolar medium /1/, which is the simplest case of a continuum with microstructure /6/. The particles of such a medium possess additional degrees of freedom, compared with the classical continuous medium, that are associated with rotation of the particle as a rigid whole. In the general case this rotation is not determined by the displacement field $\mathbf{u}$.

Let $\mathbf{x} = \mathbf{X} + \mathbf{u}$ be the particle radius-vector in the actual configuration, $\mathbf{v} = (\partial \mathbf{x}/\partial t)|_X$ the mass flow rate, and $\mathbf{F} = \nabla \mathbf{x}$ and $\mathbf{\Phi} = \nabla \boldsymbol{\varphi}$ the gradients of the displacement and the rotation such that $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, $d\boldsymbol{\varphi} = \mathbf{\Phi}d\mathbf{X}$, where

$$\boldsymbol{\varphi}^{\cdot} \equiv (\partial \boldsymbol{\varphi}/\partial t)|_X = \boldsymbol{\omega} \tag{1.1}$$

is the angular velocity vector.

If the mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{X}, t)$ are twice continuously differentiable everywhere except possibly, singular surfaces, where $\Delta = \det \mathbf{F} \neq 0$, then compatibility relationships in the variables $\mathbf{X}, t$ hold for the strain and velocity fields

$$(\partial \mathbf{F}/\partial t)|_X - \nabla \mathbf{v} = 0, \quad (\partial \mathbf{\Phi}/\partial t)|_X - \nabla \boldsymbol{\omega} = 0$$

which can be written in the form

$$\frac{\partial \mathbf{F}}{\partial t} - \operatorname{Div}(\mathbf{v} \otimes \mathbf{I}) = 0, \quad \frac{\partial \mathbf{\Phi}}{\partial t} - \operatorname{Div}(\boldsymbol{\omega} \otimes \mathbf{I}) = 0 \tag{1.2}$$

where the symbol Div denotes the divergence in the variables $\mathbf{X}$ and $\mathbf{I}$ is the unit matrix.

In the variables $\mathbf{x}, t$ the divergent form of the relationships (1.2) will be the following (div is the divergence in the variables $\mathbf{x}$):

$$\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\mathbf{F}\right)\Big|_x + \operatorname{div}\left(\frac{1}{\Delta}\mathbf{F} \otimes \mathbf{v} - \frac{1}{\Delta}\mathbf{v} \otimes \mathbf{F}^T\right) = 0$$
$$\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\mathbf{\Phi}\right)\Big|_x + \operatorname{div}\left(\frac{1}{\Delta}\mathbf{\Phi} \otimes \mathbf{v} - \frac{1}{\Delta}\boldsymbol{\omega} \otimes \mathbf{F}^T\right) = 0$$

We difine the matrices of components of the strain incompatibility tensors of the first and second kinds by the expressions

$$K_{ij}{}^m(X^a, t) = \nabla_j F_i{}^m - \nabla_i F_j{}^m, \quad H_{ij}{}^m(X^a, t) = \nabla_j \Phi_i{}^m - \nabla_i \Phi_j{}^m \tag{1.3}$$

The following relationships hold for the continuously differentiable functions $\mathbf{F}(\mathbf{X}, t)$, $\mathbf{v}(\mathbf{X}, t)$ and $\mathbf{\Phi}(\mathbf{X}, t)$ and $\boldsymbol{\omega}(\mathbf{X}, t)$:

$$K_{ij}{}^m(X^a, t) = 0, \quad H_{ij}{}^m(X^a, t) = 0 \tag{1.4}$$

In fact, by differentiating the relationship (1.2) written in a Cartesian rectangular coordinate system

$$\frac{\partial F_i{}^m(X^a, t)}{\partial t} - \frac{\partial v^m(X^a, t)}{\partial X^i} = 0, \quad \frac{\partial F_j{}^m(X^a, t)}{\partial t} - \frac{\partial v^m(X^a, t)}{\partial X^j} = 0$$

with respect to X^j and X^i, and subtracting one from the other we obtain when the smoothness of the field $v^m(X^a, t)$ is taken into account

$$\partial K_{ij}{}^m(X^a, t)/dt = 0 \tag{1.5}$$

Since the gradient is $F_i{}^m(X^a, t) = \delta_i{}^m$ when $t = 0$, and therefore, $K_{ij}{}^m(X^a, 0) = 0$, then from the uniqueness of the solution of the ordinary differential equation (1.5) with zero initial data, the first of the relationships (1.4) follows.

The validity of the second equation in (1.4) can be shown analogously if the incompatibility tensor is $H_{ij}{}^m(X^a, 0) = 0$.

In particular, it follows that relationships (1.4), which are strain compatibility relationships formulated in terms of the gradients $\mathbf{F}$ and $\mathbf{\Phi}$, are not independent equations with respect to (1.2).

As can be seen from the definition of (1.3), the matrices of the components $K_{ij}{}^m$ and $H_{ij}{}^m$ are antisymmetric in the subscripts i, j and only nine independent components exist. The matrices of these independent components, written in the form

$$B_F^{mi} = e^{ijk}\nabla_k F_j{}^m, \quad B_\Phi^{mi} = e^{ijk}\nabla_k \Phi_j{}^m$$

where e^{ijk} is the unit antisymmetric tensor, will be called matrices of the Burgers tensor components of the first and second kinds.

Therefore, the strain compatibility conditions follow from the strain and velocity compability conditions and are formulated as the conditons that the incompatibility tensors or Burgers tensor should equal zero.

Let ρ_0, ρ be the density of the body in the reference and actual configurations. The mass conservation law for a micropolar medium can be written in the form /1/

$$\rho\Delta = \rho_0, \quad \Delta = \det \mathbf{F} \tag{1.6}$$

Let $\mathbf{J} = \mathbf{J}(\mathbf{X}, t)$ and $\mathbf{J} = \mathbf{J}^T$ be a symmetric positive-definite tensor of the density of the moment of inertia of a micropolar medium. We assume for it that

$$\left.\frac{\partial \mathbf{J}}{\partial t}\right|_X = \mathbf{\Omega J} - \mathbf{J\Omega} \quad (\Omega_{ij} = e_{ikj}\omega^k) \tag{1.7}$$

This assumption is the analog of relations for the rate of change of the moment of inertia of an absolutely rigid body

$$\mathbf{J} = \int \{(\mathbf{y}\cdot\mathbf{y})\,\mathbf{I} - \mathbf{y}\otimes\mathbf{y}\}\, d^3\mathbf{y}$$

where $\mathbf{y} = \mathbf{y}(t)$ is the radius-vector of a point in the frame of reference associated with the centre of mass of the volume under consideration, and $\mathbf{I}$ is the unit tensor. If $\mathbf{R} = \mathbf{R}(t)$ is an orthogonal tensor ($\mathbf{RR}^T = \mathbf{I}$) describing the body rotation and being the analog of the axial vector $\boldsymbol{\varphi}$, then $\mathbf{y} = \mathbf{R}\mathbf{y}_0$, where $\mathbf{y}_0 = \mathbf{y}(0)$ and $\mathbf{y}^\cdot = \mathbf{R}^\cdot\mathbf{y}_0 = \mathbf{R}^\cdot\mathbf{R}^T\mathbf{R}\mathbf{y}_0 = \mathbf{\Omega y}$ and $\mathbf{J}^\cdot - \mathbf{\Omega J} + \mathbf{J\Omega} = 0$.

The divergent form of the local conservation law for the density of the moment of inertia is written in the Euler variables $\mathbf{x}$, t in the form

$$\left.\frac{\partial(\rho\mathbf{J})}{\partial t}\right|_x + \operatorname{div}(\rho\mathbf{J}\otimes\mathbf{v}) = \rho(\mathbf{\Omega J} - \mathbf{J\Omega})$$

Let $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{v}, \mathbf{X}, t)$ be the stress vector, and $\mathbf{m} = \mathbf{m}(\mathbf{v}, \mathbf{X}, t)$ the vector of a moment defined on a piecewise-smooth surface $\psi(\mathbf{X}, t) = 0$ with the normal $\mathbf{v} = \nabla\psi / |\nabla\psi|$. The vectors $\boldsymbol{\tau}$ and $\mathbf{m}$ characterise the density of interaction of parts of the body separated by the surface under consideration. The fundamental Cauchy theorem defines the Piola-Kirchhoff stress tensor of the first kind and the moments tensor

$$\boldsymbol{\tau} = \mathbf{T}(\mathbf{X}, t)\mathbf{v}, \quad \mathbf{m} = \mathbf{M}(\mathbf{X}, t)\mathbf{v}, \quad \mathbf{T} \neq \mathbf{T}^T, \quad \mathbf{M} \neq \mathbf{M}^T$$

Let $\mathbf{b}$ and $\mathbf{l}$ be the density of the mass forces and moments. Then the local momentum and angular momentum conservation laws for a micropolar medium have the following form:

$$\rho_0\frac{\partial\mathbf{v}}{\partial t} - \operatorname{Div}\mathbf{T} = \rho_0\mathbf{b}, \quad \rho_0\frac{\partial(\mathbf{J}\boldsymbol{\omega})}{\partial t} - \operatorname{Div}\mathbf{M} = \rho_0\boldsymbol{\zeta} + \rho_0\mathbf{l} \tag{1.8}$$

The vector $\boldsymbol{\zeta}$ in the angular momentum equation is an accompaniment to the Cauchy stress tensor $\Delta^{-1}\mathbf{TF}^T$ so that $\rho_0\zeta_i = e_{ijk}T^{k\alpha}F_\alpha{}^j$.

Furthermore, we will assume that the material under consideration is hyperelastic, and for simplicity, homogeneous, i.e., an elastic potential W exists such that

$$W = W(\mathbf{F}, \mathbf{\Phi}, \boldsymbol{\varphi}) \tag{1.9}$$
$$\mathbf{T} = \rho_0\frac{\partial W}{\partial\mathbf{F}}, \quad \mathbf{M} = \rho_0\frac{\partial W}{\partial\mathbf{\Phi}}, \quad \boldsymbol{\zeta} = -\frac{\partial W}{\partial\boldsymbol{\varphi}}$$

We will neglect thermal effects in the strain of such a medium. The constraints

$$W(\mathbf{QF}, \det\mathbf{Q}\cdot\mathbf{Q\Phi}, \det\mathbf{Q}\cdot\mathbf{Q}\boldsymbol{\varphi}) = W(\mathbf{F}, \mathbf{\Phi}, \boldsymbol{\varphi})$$
$$e_{ijk}\frac{\partial W}{\partial F_{k\alpha}}F_\alpha{}^j + \frac{\partial W}{\partial\varphi^i} = 0, \quad \mathbf{QQ}^T = \mathbf{I}$$

are imposed on the potential W.

The first of these is the property of objectivity — the condition of invariance of the elastic potential relative to orthogonal transformations of the actual configuration. The second of the constraints follows from the expression $\rho_0\zeta_i = e_{ijk}T^{k\alpha}F_\alpha{}^j$ and relations (1.9). Note that in the case of infinitesimal strains, the relation

$$W = W(\boldsymbol{\gamma}, \mathbf{\Phi}), \quad \boldsymbol{\gamma} = \mathbf{F} - \mathbf{e}\boldsymbol{\varphi}$$

follows from the second constraint.

Therefore, the complete set of equations for the medium being studied can be written in the form of a set of differential conservation laws (1.1), (1.2), (1.7) and (1.8) and the finite rheological relations (1.9). The set of differential equations can be represented in a Cartesian rectangular coordinate system in the form

$$\frac{\partial\varphi_\alpha{}^\circ}{\partial t} + \frac{\partial\varphi_\alpha{}^m}{\partial X^m} = f_\alpha, \quad \alpha = 1, 2, \ldots, 33, \quad m = 1, 2, 3 \tag{1.10}$$
$$\varphi_\alpha{}^\circ = \{\rho_0 v^i, \rho_0 J^{i\alpha}\omega_\alpha, \varphi^i, F_\alpha{}^i, \Phi_\alpha{}^i, J_{ij}\}$$
$$\varphi_\alpha{}^m = \{T^{im}, M^{im}, 0, v^i\delta_\alpha{}^m, \omega^i\delta_\alpha{}^m, 0\}$$
$$f_\alpha = \{\rho_0 b^i, \rho_0(l^i + \zeta^i), \omega^i, 0, 0, \Omega_{i\alpha}J_j{}^\alpha - J_i{}^\alpha\Omega_{\alpha j}\}$$

In expanded form the quasilinear system (1.10) can be written as follows

$$\frac{\partial\varphi_i}{\partial t}=\omega_i,\quad \frac{\partial F_j^{\ i}}{\partial t}-\frac{\partial v^i}{\partial X^j}=0,\quad \frac{\partial\Phi_j^{\ i}}{\partial t}-\frac{\partial\omega^i}{\partial X^j}=0 \tag{1.11}$$
$$\frac{\partial v^i}{\partial t}-\frac{\partial^2W}{\partial F_m^{\ i}\partial F_n^{\ j}}\frac{\partial F_n^{\ j}}{\partial X^m}-\frac{\partial^2W}{\partial F_m^{\ i}\partial\Phi_n^{\ j}}\frac{\partial\Phi_n^{\ j}}{\partial X^m}-\frac{\partial^2W}{\partial F_m^{\ i}\partial\varphi_j}\frac{\partial\varphi_j}{\partial X^m}=b_i$$
$$J_i^{\ a}\frac{\partial\omega_a}{\partial t}-\frac{\partial^2W}{\partial\Phi_m^{\ i}\partial F_n^{\ j}}\frac{\partial F_n^{\ j}}{\partial X^m}-\frac{\partial^2W}{\partial\Phi_m^{\ i}\partial\Phi_n^{\ j}}\frac{\partial\Phi_n^{\ j}}{\partial X^m}-$$
$$-\frac{\partial^2W}{\partial\Phi_m^{\ i}\partial\varphi_j}\frac{\partial\varphi_j}{\partial X^m}=l_i+\zeta_i+e_{iab}J_j^{\ a}\omega^b\omega^j$$
$$\frac{\partial J_{ij}}{\partial t}=e_{iba}\omega^bJ_j^{\ a}-e_{abj}J_i^{\ a}\omega^b$$

For the sequel it is convenient to use (1.11) in the matrix form

$$\frac{\partial u_\alpha}{\partial t}+B_{\alpha\beta}^m(u_\gamma)\frac{\partial u_\beta}{\partial X^m}=f_\alpha(u_\gamma) \tag{1.12}$$
$$u_\alpha=\{v_i,\ \omega_i,\ \varphi_i,\ F^i{}_j,\ \Phi^i{}_j,\ J_{ij}\}$$
$$\alpha,\beta,\gamma=1,2,\ldots,33;\ i,j,m=1,2,3$$

2. Hyperbolicity conditions. Let us investigate the question of the hyperbolicity of (1.12) which is closely associated with the correctness of the Cauchy problem. By definition /8/, system (1.12) is hyperbolic if all the eigennumbers of the matrix $\nu_mB_{\alpha\beta}{}^m$, where ν_m is an arbitrary unit vector, are real and the number of their corresponding linearly independent left eigenvectors of this matrix equals the number of equations in (1.12) for any given ν_m.

We will obtain an equation for the eigennumbers, or in other words, for the velocities of propagation of the characteristic surfaces. Let $\psi(\mathbf{X},t)=0$ be the equation of such a surface, which is simultaneously a surface of possible weak discontinuities /8/, $c=-\partial\psi/\partial t/|\nabla\psi|$, $\boldsymbol{\nu}=\nabla\psi/|\nabla\psi|$ is the velocity of propagation and the unit normal. Then, using the geometric and kinematic compatibility conditions /9/, we obtain on the surface of weak discontinuity

$$c\left[\frac{\partial\varphi_i}{\partial\nu}\right]=0,\quad c\left[\frac{\partial F_j^{\ i}}{\partial\nu}\right]+\left[\frac{\partial v^i}{\partial\nu}\right]\nu_j=0 \tag{2.1}$$
$$c\left[\frac{\partial\Phi_j^{\ i}}{\partial\nu}\right]+\left[\frac{\partial\omega^i}{\partial\nu}\right]\nu_j=0$$
$$c\left[\frac{\partial v^i}{\partial\nu}\right]+\nu_m\frac{\partial^2W}{\partial F_m^{\ i}\partial F_n^{\ j}}\left[\frac{\partial F_n^{\ j}}{\partial\nu}\right]+\nu_m\frac{\partial^2W}{\partial F_m^{\ i}\partial\Phi_n^{\ j}}\left[\frac{\partial\Phi_n^{\ j}}{\partial\nu}\right]+$$
$$\nu_m\frac{\partial^2W}{\partial F_m^{\ i}\partial\varphi_j}\left[\frac{\partial\varphi_j}{\partial\nu}\right]=0$$
$$cJ_i^{\ a}\left[\frac{\partial\omega_a}{\partial\nu}\right]+\nu_m\frac{\partial^2W}{\partial\Phi_m^{\ i}\partial F_n^{\ j}}\left[\frac{\partial F_n^{\ j}}{\partial\nu}\right]+\nu_m\frac{\partial^2W}{\partial\Phi_m^{\ i}\partial\Phi_n^{\ j}}\left[\frac{\partial\Phi_n^{\ j}}{\partial\nu}\right]+$$
$$\nu_m\frac{\partial^2W}{\partial\Phi_m^{\ i}\partial\varphi_j}\left[\frac{\partial\varphi_j}{\partial\nu}\right]=0,\quad c\left[\frac{\partial J_{ij}}{\partial\nu}\right]=0$$

where $[\partial u_\alpha/\partial\nu]$ is the jump in the normal derivative of the solution defined by the formula $\partial u_\alpha/\partial\nu=\nu_i\partial u_\alpha/\partial X^i$.

The non-trivial solution $[\partial u_\alpha/\partial\nu]$ of the homogeneous linear system (2.1) exists if and only if the determinant of the coefficient matrix equals zero, i.e.,

$$\det\|-c\delta_{\alpha\beta}+\nu_mB_{\alpha\beta}{}^m\|=0 \tag{2.2}$$

It is seen directly from (2.1) that $c=0$ is a multiple root of (2.2). If $c\neq0$, then it follows from (2.1)

$$\det\left\|c^2\delta_{\lambda\mu}-\nu_m\frac{\partial^2W}{\partial p_m^{\ \lambda}\partial p_n^{\ \mu}}\nu_n\right\|=0 \tag{2.3}$$
$$p_m{}^\lambda=\{F_m{}^i,\Phi_m{}^i\},\quad \lambda,\mu=1,2,\ldots,6;\quad i,m,n=1,2,3$$

whose solvability condition is

$$\left(\nu_m\frac{\partial^2W}{\partial p_m^{\ \lambda}\partial p_n^{\ \mu}}\nu_n\right)a^\lambda a^\mu>0 \tag{2.4}$$

for all propagation directions determined by the normal ν_m, and all $a^\lambda\neq0$.

The constraint (2.4) on the form of the elastic potential W is a necessary condition for hyperbolicity and the analog of the *SE*-condition in the non-linear theory of elasticity /4, 7/.

If inequality (2.4) holds, then system (1.11) possesses real propagation velocities of the characteristic surfaces. But it is impossible to say anything about the existence and linear independence of the eigenvectors of the matrix $\nu_mB_{\alpha\beta}{}^m$ by using only (2.4).

We will formulate a sufficient condition for hyperbolicity. To do this, we will reduce system (1.10) to symmetric form as in /2, 3/.

Direct substitution confirms the validity of the equations (E is the total energy density)

$$v_\alpha f_\alpha = \rho_0(b^i v_i + l^i \omega_i), \quad v_\alpha d\varphi_\alpha{}^\circ = d\,(\rho_0 E)$$
$$v_\alpha d\varphi_\alpha{}^m = d\,(T^{im} v_i + M^{im}\omega_i)$$
$$(E = W + v_i v^i/2 + J^{ij}\,\omega_i\omega_j/2) \tag{2.5}$$

Here

$$v_\alpha = \{v_i,\ \omega_i,\ -\rho_0\zeta_i,\ T_{ij},\ M_{ij},\ -{}^1/_2\rho_0\omega_i\omega_j\} \tag{2.6}$$

Rewriting the second and third equations in (2.5) in the form

$$dL^\circ = d\,(v_\alpha\varphi_\alpha{}^\circ - \rho_0 E) = \varphi_\alpha{}^\circ dv_\alpha$$
$$dL^m = d\,(v_\alpha\varphi_\alpha{}^m - T^{im}v_i - M^{im}\omega_i) = \varphi_\alpha{}^m\,dv_\alpha$$

we can write system (1.10) in the form

$$\frac{\partial}{\partial t}\frac{\partial L^\circ}{\partial v_\alpha} - \frac{\partial}{\partial X^m}\frac{\partial L^m}{\partial v_\alpha} = f_\alpha \tag{2.7}$$
$$\left(\varphi_\alpha{}^\circ = \frac{\partial L^\circ}{\partial v_\alpha},\ \varphi_\alpha{}^m = \frac{\partial L^m}{\partial v_\alpha}\right)$$

If we use the notation

$$L_{\alpha\beta}{}^\circ = \partial^2 L^\circ/\partial v_\alpha\partial v_\beta, \quad L_{\alpha\beta}{}^\circ = L_{\beta\alpha}{}^\circ$$
$$L_{\alpha\beta}{}^m = \partial^2 L^m/dv_\alpha dv_\beta, \quad L_{\alpha\beta}{}^m = L_{\beta\alpha}{}^m$$

then from (2.7) we obtain the desired symmetric system

$$\overset{\circ}{L}_{\alpha\beta}\frac{\partial v_\beta}{\partial t} - \overset{m}{L}_{\alpha\beta}\frac{\partial v_\beta}{\partial X^m} = f_\alpha \tag{2.8}$$

with the vector of the solution v_α whose relation to the vector u_α is determined by (2.6).

We now note that if the matrix $L_{\alpha\beta}{}^\circ$ is positive-definite, then a non-degenerate transformation exists that simultaneously reduces the two symmetric matrices $L_{\alpha\beta}{}^\circ$ and $\nu_m L_{\alpha\beta}{}^m$ to diagonal form, where ν_m is the normal vector. In this case system (2.8) is known to be hyperbolic.

The positive-definiteness of $L_{\alpha\beta}{}^\circ$ is equivalent to the convexity of the function L° in the arguments v_β. The Legendre transformation of an arbitrary convex function $M\,(v_\alpha)$ has the form $H\,(z_\alpha) = z_\alpha v_\alpha - M\,(v_\alpha)$, $z_\alpha = \partial M/\partial v_\alpha$, and is itself a convex function. For the function L° it equals $H\,(\varphi_\alpha{}^\circ) = \rho_0 E$. Because $\rho_0 > 0$ and in view of the positive-definiteness of the tensor J^{ij}, the total energy E will be convex function provided the elastic potential $W = W\,(F_a{}^i,\ \Phi_a{}^i,\ \varphi^i)$ is convex, i.e.,

$$\frac{\partial^2 W}{\partial\pi_\varkappa\partial\pi_\mu}\lambda^\varkappa\lambda^\mu > 0, \quad \forall\lambda^\varkappa \neq 0 \tag{2.9}$$
$$\pi_\varkappa = \{F_a{}^i,\ \Phi_a{}^i,\ \varphi^i\}$$

It also follows from the convexity of $W\,(\pi_\varkappa)$ that the transformation of the initial system (1.12) into the symmetric system (2.8), associated with replacing the solution vector $u_\alpha = \{v_i, \omega_i, \varphi_i, F_j{}^i, \Phi_j{}^i, J_{ij}\}$ by the vector (2.6), is non-degenerate, i.e.,

$$\det\|\,\partial u_\alpha/\partial v_\beta\,\| = \det\|\,\partial u_\alpha/\partial\varphi_\gamma{}^\circ\,\|\,\det\|\,\partial\varphi_\gamma{}^\circ/\partial v_\beta\,\| \neq 0$$

Comparing the sufficient condition (2.9) with the necessary condition (2.4), it can be seen that (2.9) is known to be stronger than (2.4).

3. Estimation of the growth of the solutions of the Cauchy problem. Let a symmetric hyperbolic system (2.8) be defined in a four-dimensional open domain Ω of the variables $\mathbf{X}, t$. The boundary $\partial\Omega$ consists of a three-dimensional domain $\omega(0)$ in the plane $t = 0$. and of a piecewise-smooth surface $\Gamma\,(\mathbf{X},\ t) = 0$ located for $t > 0$. We shall assume that $\nabla\Gamma \neq 0$, within the domain $\Gamma < 0$ and outside it $\Gamma > 0$. Let us also assume that

$$v_\gamma\,(\mathbf{X},\ t),\ L_{\alpha\beta}{}^\circ\,(v_\gamma\,(X,\ t))$$
$$L_{\alpha\beta}{}^m\,(v_\gamma\,(\mathbf{X},\ t)),\ f_\alpha\,(v_\gamma\,(\mathbf{X},\ t)) \in C_1\,(\overline{\Omega}) \tag{3.1}$$

At any point $(\mathbf{X},\ t) \in \Gamma$ let the normal to the surface $\Gamma = 0$ lie within the cone of normals to the characteristic surface corresponding to the maximum, in absolute value, $c = c\,(v\ (\mathbf{X},\ t),\ \boldsymbol{\nu})$, $\boldsymbol{\nu} = \nabla\Gamma/\,|\,\nabla\Gamma\,|$, i.e., the Hamilton-Jacobi inequality holds /8/

$$G\,(\mathbf{X}, t) \geqslant |\,c\,(v_\gamma(\mathbf{X}, t), \boldsymbol{\nu})\,|, \quad G = \frac{1}{|\,\nabla\Gamma\,|}\frac{\partial\Gamma}{\partial t} > 0 \tag{3.2}$$

Then for any other solution $v_\gamma'\,(\mathbf{X},\ t) \in C_1\,(\overline{\Omega})$ the following estimate holds

$$\|\,\mathbf{v} - \mathbf{v}'\,\|_t{}^2 \leqslant Ne^{tN}\,\|\,\mathbf{v} - \mathbf{v}'\,\|_0{}^2, \quad N = \mathrm{const} > 0 \tag{3.3}$$

$$\| \mathbf{v} \|_t^2 = \int\limits_{\omega(t)} v_\alpha \overset{\circ}{L}_{\alpha\beta} v_\beta \, d\omega$$

where $\omega(t)$ is the section $t = \mathrm{const}$ of the domain Ω.

To prove inequality (3.3), we will consider the solutions $\mathbf{v}$ and $\mathbf{v}'$ of two Cauchy problems for system (2.8) with the initial data

$$\mathbf{v}(\mathbf{X}, 0) = \mathbf{v}_0(\mathbf{X}), \quad \mathbf{v}'(\mathbf{X}, 0) = \mathbf{v}_0'(\mathbf{X}), \quad \mathbf{X} \in \omega(0)$$

We subtract from the (2.8) corresponding to $\mathbf{v}'$ Eq.(2.8) for the solution $\mathbf{v}$ and we use the notation $\mathbf{w}(\mathbf{X}, t) = \mathbf{v}'(\mathbf{X}, t) - \mathbf{v}(\mathbf{X}, t)$. Multiplying the result scalarly by $2\mathbf{w}$ and using the symmetry of the matrices $\mathbf{L}^\circ$ and $\mathbf{L}^m$, we arrive at the relation

$$\frac{\partial}{\partial t}(\mathbf{w}\mathbf{L}^\circ\mathbf{w}) + \frac{\partial}{\partial X^m}(\mathbf{w}\mathbf{L}^m\mathbf{w}) = \mathbf{w}\mathbf{R}\mathbf{w} \tag{3.4}$$

$$\mathbf{R} = 2\mathbf{R}_1 + \partial\mathbf{L}^\circ(\mathbf{v}(\mathbf{X}, t))/\partial t + \partial\mathbf{L}^m(\mathbf{v}(\mathbf{X}, t))/\partial X^m.$$

$$\mathbf{R}_1\mathbf{w} = \mathbf{f}(\mathbf{v}') - \mathbf{f}(\mathbf{v}) + (\mathbf{L}^\circ(\mathbf{v}') - \mathbf{L}^\circ(\mathbf{v}))\frac{\partial\mathbf{v}'}{\partial t} + (\mathbf{L}^m(\mathbf{v}') - \mathbf{L}^m(\mathbf{v}))\frac{\partial\mathbf{v}'}{\partial X^m}$$

Integrating (3.4) over the domain Ω_t which is the part of the domain included between the planes $t = 0$ and $t = \mathrm{const} > 0$, and applying Gauss's theorem, we obtain

$$\int\limits_{\omega(t)} \mathbf{w}\mathbf{L}^\circ\mathbf{w}\, d\omega - \int\limits_{\omega(0)} \mathbf{w}\mathbf{L}^\circ\mathbf{w}\, d\omega + \int\limits_{\Gamma(t)} \mathbf{w}(G\mathbf{L}^\circ + \nu_m\mathbf{L}^m)\mathbf{w}\, d\Gamma = \int\limits_0^t \int\limits_{\omega(t')} \mathbf{w}\,\mathbf{R}\mathbf{w}\, d\omega\, dt' \tag{3.5}$$

where $\Gamma(t)$ is the part of the boundary Γ between the planes $t = 0$ and $t = \mathrm{const}$.

It is asserted that

$$\mathbf{w}(G\mathbf{L}^\circ + \nu_m\mathbf{L}^m)\mathbf{w} > 0 \tag{3.6}$$

Indeed, as has already been mentioned in Sect.2, a matrix $\mathbf{S} = \mathbf{S}(\mathbf{v})$ $\det\mathbf{S} \neq 0$ exists such that $\mathbf{L}^\circ = \mathbf{S}\Lambda\mathbf{S}^T$, $\nu_m\mathbf{L}^m = \mathbf{S}\mathbf{D}\mathbf{S}^T$, $\Lambda > 0$, where Λ and $\mathbf{D}$ are diagonal matrices. And it follows from the characteristic equation $\det\| -c\mathbf{L}^\circ + \nu_m\mathbf{L}^m \| = 0$ that $D_{\alpha\alpha} = c_\alpha\Lambda_{\alpha\alpha}$ (not summed over α). But then, by using the notation $\mathbf{w}^* = \mathbf{S}^T\mathbf{w}$ we obtain, when (3.2) is taken into account,

$$\mathbf{w}(G\mathbf{L}^\circ + \nu_m\mathbf{L}^m)\mathbf{w} = \mathbf{w}^*(G\Lambda + \mathbf{D})\mathbf{w}^* = \sum_\alpha (G + c_\alpha) w_\alpha^* \Lambda_{\alpha\alpha} w_\alpha^* > 0$$

Furthermore, the assumptions of smoothness (3.1) and condition (2.9) of the positive-definiteness of $\mathbf{L}^\circ$ assure the inequality

$$m|\mathbf{w}|^2 \leqslant \mathbf{w}\mathbf{L}^\circ\mathbf{w} \leqslant M|\mathbf{w}|^2, \quad m > 0$$

$$|\mathbf{w}\mathbf{R}\mathbf{w}| \leqslant N_1|\mathbf{w}|^2 \leqslant N\mathbf{w}\mathbf{L}^\circ\mathbf{w}, \quad N = N_1/m$$

which enable us, together with (3.6), to obtain from (3.5)

$$\|\mathbf{w}\|_t^2 \leqslant \|\mathbf{w}\|_0^2 + N\int\limits_0^t \|\mathbf{w}\|_\tau^2 d\tau$$

Hence, by using the lemma on an integral inequality (/10/, p.123), we obtain the estimate (3.3).

The unqiueness of the solution of the Cauchy problem in the class C_1 within any domain subject to the condition (3.2) follows directly from (3.3).

By demanding that the solutions, the coefficients, and the right side of (2.8) should be continuous, and using the continued system, the growth of the first and higher derivatives of the solution can be estimated analogously.

4. The transport equation. Consider the equation of the change with time of the amplitude of weak discontinuities propagating over the moving $(c \neq 0)$ characteristic surfaces. To derive it, we differentiate the symmetric system (2.8) with respect to time for $\mathbf{X} = \mathrm{const}$ Using the notation

$$q_\beta = \frac{\partial v_\beta}{\partial t}, \quad p_\beta{}^m = \frac{\partial v_\beta}{\partial X^m}, \quad \overset{\circ}{L}_{\alpha\beta\gamma} = \frac{\partial \overset{\circ}{L}_{\alpha\beta}}{\partial v_\gamma}$$

$$L^m_{\alpha\beta\gamma} = \frac{\partial L^m_{\alpha\beta}}{\partial v_\gamma}, \quad f_{\alpha\beta} = \frac{\partial f_\alpha}{\partial v_\beta}$$

and the fact that

$$\partial^2 v_\beta/\partial X^i\partial t = \partial^2 v_\beta/\partial t\partial X^i$$

for $v_\beta \in C_2$, as we shall indeed assume outside the surface of discontinuity, we obtain

$$\overset{\circ}{L}_{\alpha\beta}\frac{\partial q_\beta}{\partial t} + L^m_{\alpha\beta}\frac{\partial q_\beta}{\partial X^m} = f_{\alpha\beta} - \overset{\circ}{L}_{\alpha\beta\gamma}q_\beta q_\gamma - L^m_{\alpha\beta\gamma}q_\gamma p_\beta{}^m \tag{4.1}$$

On the singular surface $\psi^{(\lambda)}(\mathbf{X}, t) = 0$, corresponding to the eigen number $c^{(\lambda)}$, we obtain from (4.1) the equation for the jumps

$$L^{\circ}_{\alpha\beta}\left[\frac{\partial q_\beta}{\partial t}\right] + L^{m}_{\alpha\beta}\left[\frac{\partial q_\beta}{\partial X^m}\right] = f_{\alpha\beta} - L_{\alpha\beta\gamma}[q_\beta q_\gamma] - L^{m}_{\alpha\beta\gamma}[q_\gamma p_\beta{}^m] \tag{4.2}$$

We will use the compatability conditions /9/ on the surface $\psi^{(\lambda)} = 0$ of the weak discontinuity

$$\left[\frac{\partial q_\beta}{\partial t}\right] = -c^{(\lambda)}\left[\frac{\partial q_\beta}{\partial \nu}\right] + \frac{\delta[q_\beta]}{\delta t^{(\lambda)}}$$
$$\left[\frac{\partial q_\beta}{\partial X^i}\right] = \left[\frac{\partial q_\beta}{\partial \nu}\right]\nu_i + g^{AB}\frac{\partial[q_\beta]}{\partial y^A_{(\lambda)}}\frac{\partial X^i}{\partial y^B_{(\lambda)}}$$

where g^{AB} is the metric tensor of the curvilinear coordinate system $y^A_{(\lambda)}$ $(A = 1,2)$ introduced on the surface $\psi^{(\lambda)} = 0$ in such a manner that $\partial\mathbf{X}/\partial t\,|_{y^A_{(\lambda)}} = c^{(\lambda)}\boldsymbol{\nu}$, while the quantity $\delta q_\beta/\delta t^{(\lambda)} \equiv \partial q_\beta/\partial t\,|_{y^A_{(\lambda)}}$ is the derivative along the bicharacteristic. Taking account of the relations

$$[q_\gamma q_\beta] = q_\beta^-[q_\gamma] + q_\gamma^-[q_\beta] + [q_\gamma][q_\beta]$$
$$[p_\beta{}^m] = -\frac{\nu_m}{c^{(\lambda)}}[q_\beta] \tag{4.3}$$

where the quantities directly ahead of the wave front are marked with a minus sign, we reduce (4.2) to the form

$$(-c^{(\lambda)}L^{\circ}_{\alpha\beta} + \nu_m L^{m}_{\alpha\beta})\left[\frac{\partial q_\beta}{\partial \nu}\right] + L^{\circ}_{\alpha\beta}\frac{\delta[q_\beta]}{\delta t^{(\lambda)}} +$$
$$g^{AB}L^{m}_{\alpha\beta}\frac{\partial X^m}{\partial y^B_{(\lambda)}}\frac{\partial[q_\beta]}{\partial y^A_{(\lambda)}} = a_{\alpha\beta}[q_\beta] - b_{\alpha\beta\gamma}[q_\beta][q_\gamma] \tag{4.4}$$
$$a_{\alpha\beta} = f_{\alpha\beta} - 2L^{\circ}_{\alpha\beta\gamma}q_\gamma^- + \frac{\nu_m}{c^{(\lambda)}}L^{m}_{\alpha\beta\gamma}q_\gamma^- - L^{m}_{\alpha\beta\gamma}p_\gamma^{m-}$$
$$b_{\alpha\beta\gamma} = L^{\circ}_{\alpha\beta\gamma} - \frac{\nu_m}{c^{(\lambda)}}L^{m}_{\alpha\beta\gamma}$$

Equation (4.4) contains jumps in both the first and second derivatives of the vector of the solution of (2.8). To get rid of $[\partial q_\beta/\partial\nu]$, we multiply (4.4) on the left by $\omega_{\lambda\alpha}$ where $\omega_{\lambda\alpha}$ are null-vectors of the symmetric matrix $(-c^{(\lambda)}L_{\alpha\beta}{}^{\circ} + \nu_m L_{\alpha\beta}{}^m)$ corresponding to the velocities of propagation $c^{(\lambda)}$ of the characteristic surface $\psi^{(\lambda)} = 0$. We consequently obtain (here and henceforth we do not sum over λ)

$$\omega_{\lambda\alpha}L^{\circ}_{\alpha\beta}\frac{\delta[q_\beta]}{\delta t^{(\lambda)}} + g^{AB}\omega_{\lambda\alpha}L^{m}_{\alpha\beta}\frac{\partial X^m}{\partial y^B_{(\lambda)}}\frac{\partial[q_\beta]}{\partial y^A_{(\lambda)}} = \omega_{\lambda\alpha}a_{\alpha\beta}[q_\beta] - \omega_{\lambda\alpha}b_{\alpha\beta\gamma}[q_\beta][q_\gamma] \tag{4.5}$$

Let the multiplicity of the eigen number $c^{(\lambda)}$ be m and less then the number $m^* = 33$ of the equations of system (2.8), as follows from Sect.2. Relations (4.5) then represent m equations for m^* unknowns $[q_\beta]$. In order to eliminate part of the unknowns, we use the equation

$$(-c^{(\lambda)}L_{\alpha\beta}{}^{\circ} + \nu_m L_{\alpha\beta}{}^m)\ [q_\beta] = 0 \tag{4.6}$$

that follows from (2.8) and the relation $[\partial v_\beta/\partial\nu] = -[q_\beta]/c^{(\lambda)}$. To do this, we note that the coefficient matrix (4.6) is symmetric, i.e., its left null-vectors are simultaneously also right null-vectors.

Furthermore, it follows from the hyperbolicity of (2.8) that a matrix whose rows are the null vectors $\omega_{\lambda\alpha}$ will be non-degenerate. Therefore, the general solution of (4.6) can be represented in the form

$$[q_\beta] = Q_\mu\omega_{\mu\beta};\ \mu = 1, 2, \ldots, m,\ \beta = 1, 2, \ldots, m^* \tag{4.7}$$

Substituting (4.7) into (4.5), we arrive at a system of m equations in m unknowns

$$A_{\lambda\mu}\frac{\delta Q_\mu}{\delta t^{(\lambda)}} + B^A_{\lambda\mu}\frac{\partial Q_\mu}{\partial y^A_{(\lambda)}} = C_{\lambda\mu}Q_\mu + D_{\lambda\mu\zeta}Q_\mu Q_\zeta \tag{4.8}$$
$$\lambda,\ \mu,\ \zeta = 1, 2, \ldots, m,\ A = 1, 2,\ \alpha,\ \beta,\ \gamma = 1, 2, \ldots, m^*$$

Here

$$A_{\lambda\mu} = \omega_{\lambda\alpha}\omega_{\mu\beta}L^{\circ}_{\alpha\beta},\quad A_{\lambda\mu} = A_{\mu\lambda}$$
$$B^A_{\lambda\mu} = \omega_{\lambda\alpha}\omega_{\mu\beta}L^{m}_{\alpha\beta}g^{AB}\frac{\partial X^m}{\partial y^B_{(\lambda)}},\quad B^A_{\lambda\mu} = B^A_{\mu\lambda}$$

$$C_{\lambda\mu} = a_{\alpha\beta}\omega_{\lambda\alpha}\omega_{\mu\beta} - \omega_{\lambda\alpha}L_{\alpha\beta}\frac{\delta\omega_{\mu\beta}}{\delta t^{(\lambda)}} - \omega_{\lambda\alpha}L_{\alpha\beta}^{m}\frac{\partial X^m}{\partial y_{(\lambda)}^{B}}\frac{\partial\omega_{\mu\beta}}{\partial y_{(\lambda)}^{A}}g^{AB}$$

$$D_{\lambda\mu\zeta} = \omega_{\lambda\alpha}\omega_{\mu\beta}\omega_{\zeta\gamma}b_{\alpha\beta\gamma}$$

To derive the equations that describes the change with time of the intensity of the weak discontinuity on a material surface ($c \equiv 0$), the continued system obtained from (2.8) by the action of the operator $\nu_i\partial/\partial X^i$ should be considered. Taking into account the formula $\delta\nu_i/\delta t = \nu_i\partial\nu_i/\partial y^A = 0$ for $c = 0$, relationships (4.3), the compatibility conditions, and the representations (4.7) for the jump in the normal derivatives, we again arrive at system (4.8) with the coefficient matrices of the right side

$$C_{\lambda\mu} = \omega_{\lambda\alpha}\omega_{\mu\beta}\{f_{\alpha\beta} - \overset{\circ}{L}_{\alpha\beta\gamma}q_\gamma - L_{\alpha\beta\gamma}^{m}(p_\gamma^{m-} + \nu_i\nu_m p_\gamma^{i-})\} -$$
$$\omega_{\lambda\alpha}\overset{\circ}{L}_{\alpha\beta}\frac{\delta\omega_{\mu\beta}}{\delta t} - \omega_{\lambda\alpha}L_{\alpha\beta}^{m}\frac{\partial X^m}{\partial y_{(\lambda)}^{B}}g^{AB}\frac{\partial\omega_{\mu\beta}}{\partial y_{(\lambda)}^{A}}$$
$$D_{\lambda\mu\zeta} = -\omega_{\lambda\alpha}\omega_{\mu\beta}\omega_{\zeta\gamma}L_{\alpha\beta\gamma}^{m}\nu_m$$

It follows from the non-linearity of the right side of (4.8) that the intensity of the weak discontinuity in the solution of the equations of the dynamics of the medium under consideration can become infinite in a finite time interval, i.e., the weak discontinuity is converted into a shock or a contact discontinuity.

5. Strong discontinuities.

We will formulate relations on strong discontinuities in a non-linearly-elastic micropolar medium. Let $\psi(X, t) = 0$ be the equation of the surface of a strong discontinuity, $c = -\partial\psi/\partial t/\,|\nabla\psi|$, $\nu = \nabla\psi/\,|\nabla\psi|$ is the velocity of motion and the unit normal, respectively. Then, for the divergent system (1.10), the following relationships /8/ hold on the strong discontinuity:

$$-c\,[\varphi_\alpha{}^\circ] + \nu_m\,[\varphi_\alpha{}^m] = 0$$

Using the expresion for $\varphi_\alpha{}^\circ$, $\varphi_\alpha{}^m$, we obtain

$$\begin{aligned} &\rho_0 c\,[\mathbf{v}] + [\mathbf{T}]\,\boldsymbol{\nu} = 0, \quad c\,[\boldsymbol{\varphi}] = 0\\ &\rho_0 c\,[\mathbf{J}\boldsymbol{\omega}] + [\mathbf{M}]\,\boldsymbol{\nu} = 0, \quad c\,[\mathbf{J}] = 0\\ &c\,[\mathbf{F}] + [\mathbf{v}]\otimes\boldsymbol{\nu} = 0, \quad c\,[\boldsymbol{\Phi}] + [\boldsymbol{\omega}]\otimes\boldsymbol{\nu} = 0 \end{aligned} \tag{5.1}$$

It hence follows that on the shock $(c \neq 0)$

$$\begin{aligned} &[\mathbf{F}] = \mathbf{h}\otimes\boldsymbol{\nu}, \quad [\boldsymbol{\Phi}] = \mathbf{k}\otimes\boldsymbol{\nu}, \quad \mathbf{h} = -c^{-1}[\mathbf{v}], \quad \mathbf{k} = -c^{-1}[\boldsymbol{\omega}],\\ &[\boldsymbol{\varphi}] = 0, \quad [\mathbf{J}] = 0 \end{aligned} \tag{5.2}$$

and the first two equations of (5.1) take the form

$$[\mathbf{T}]\,\boldsymbol{\nu} - \rho_0 c^2\mathbf{h} = 0, \quad [\mathbf{M}]\,\boldsymbol{\nu} - \rho_0 c^2\mathbf{k} = 0 \tag{5.3}$$

If the state of the medium $\mathbf{F}^\circ$, $\boldsymbol{\Phi}^\circ$, $\boldsymbol{\varphi}^\circ$, $\mathbf{J}^\circ$ is known ahead of the shock front with the given velocity of motion c, then for the solution behind the front to be unique it is necessary that the shock velocity differ from the velocity of the characteristic surface. Indeed, by taking account of (5.2) and considering the relations (5.3) as a system of equations in the quantities $\mathbf{h}$ and $\mathbf{k}$

$$\begin{aligned} &f(\mathbf{h}, \mathbf{k}) = \mathbf{T}(\mathbf{F}^\circ + \mathbf{h}\otimes\boldsymbol{\nu}, \boldsymbol{\Phi}^\circ + \mathbf{k}\otimes\boldsymbol{\nu}, \boldsymbol{\varphi}^\circ)\,\boldsymbol{\nu} -\\ &\quad \mathbf{T}(\mathbf{F}^\circ, \boldsymbol{\Phi}^\circ, \boldsymbol{\varphi}^\circ)\,\boldsymbol{\nu} - \rho_0 c^2\mathbf{h} = 0\\ &g(\mathbf{h}, \mathbf{k}) = \mathbf{M}(\mathbf{F}^\circ + \mathbf{h}\otimes\boldsymbol{\nu}, \boldsymbol{\Phi}^\circ + \mathbf{k}\otimes\boldsymbol{\nu}, \boldsymbol{\varphi}^\circ)\,\boldsymbol{\nu} -\\ &\quad \mathbf{M}(\mathbf{F}^\circ, \boldsymbol{\Phi}^\circ, \boldsymbol{\varphi}^\circ)\,\boldsymbol{\nu} - \rho_0 c^2\mathbf{k} = 0 \end{aligned}$$

we obtain

$$\frac{\partial(\mathbf{f}, \mathbf{g})}{\partial(\mathbf{h}, \mathbf{k})} = \det\left\| c^2\delta_{\alpha\beta} - \nu_m\frac{\partial^2 W}{\partial p_m{}^\alpha\partial p_n{}^\beta}\nu_n\right\| \neq 0$$
$$p_m{}^\alpha = \{F_m{}^i, \Phi_m{}^i\};\ \alpha, \beta = 1, 2, \ldots, 6;\ m, n, i = 1, 2, 3$$

On the contact discontinuity $(c \equiv 0)$, the continuity of the stress and moment vectors

$$[\mathbf{T}]\,\boldsymbol{\nu} = 0, \ [\mathbf{M}]\,\boldsymbol{\nu} = 0$$

follows from (5.1), as does the continuity of the velocities $[\mathbf{v}] = [\boldsymbol{\omega}] = 0$. The latter relationship is a result of the assumption on the uniqueness of the mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{X}, t)$. In the more general case, the compatibility equations (1.2) become inhomogeneous, with a singular right side, and the velocity vectors can undergo a discontinuity on the contact surface.

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ON THE STABILITY OF ONE-DIMENSIONAL STATIONARY SOLUTIONS OF HYPERBOLIC SYSTEMS OF DIFFERENTIAL EQUATIONS CONTAINING POINTS AT WHICH ONE CHARACTERISTIC VELOCITY BECOMES ZERO*

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The stability of the stationary solutions of hyperbolic systems of partial differential equations containing a point at which one of the characteristic velocities becomes zero, is investigated. The functions sought are assumed to be time and coordinate dependent, and their number is arbitrary.

The study of stability carried out below is based on the results obtained in /1, 2/, according to which the behaviour of the unsteady perturbations near the critical point is described by a single non-linear partial differential equation irrespective of the number of equations in the initial system. The equation is written in terms of a function analogous to the Riemann invariant connected with the vanishing characteristic velocity.

The equation is used below to examine all possible cases of continuous solutions of an arbitrary hyperbolic system of equations with continuous and discontinuous right-hand sides, and conditions are formulated under which the growth of perturbations near the critical point at which one of the characteristic velocities becomes zero, leads to the instability of the whole solution in toto. The investigation is carried out taking into account the onset and development of the perturbations connected with other characteristic velocities which have a constant sign within the region considered.

1. Let us consider a hyperbolic system containing an arbitrary number of equations the unknown functions of which depend on the spatial coordinate x and the time t

$$l_j^i(u_k, x)\left[\frac{\partial u_j}{\partial t} + c^i(u_k, x)\frac{\partial u_j}{\partial x}\right] = f^i(u_k, x) \tag{1.1}$$

System (1.1) is written in the characteristic form, $c^i(u_k, x)$ are the characteristic velocities of the system, and repeated lower Latin indices denote summation from 1 to n.

The elements of the matrix l_j^i and the function c^i are assumed to be continuous and differentiable functions of their arguments, and the right-hand sides of (1.1) are assumed to be piecewise continuous and may have first-order discontinuities in some planes $x = \text{const}$. We shall assume that the first-order partial derivatives of $f^i(u_k, x)$ with respect to all arguments exist and are continuous wherever $f^i(u_k, x)$ are defined, except at the points belonging to the surfaces of discontinuity of these functions.

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